Real-world and risk-neutral probabilities in the regulation on the transparency of structured products

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January 2015

Abstract

The price of derivatives (and hence of structured products) can be calculated as the discounted value of expected future payoffs, assuming standard hypotheses on frictionless and complete markets and on the type of stochastic processes for the price of the underlying. However, the probabilities used in the pricing process do not represent "real" probabilities of future events, because they are based on the assumption that market participants are risk-neutral. This paper reviews the relevant mathematical finance literature, and clarifies that the risk-neutrality hypothesis is acceptable for pricing, but not to forecast the future value of an asset. Therefore, we argue that regulatory initiatives that mandate intermediaries to give retail investors information on the probability that, at a future date, the value of a derivative will be higher or lower than a given threshold (so-called "probability scenarios") should explicitly reference probabilities that take into account the risk premium (so-called "real-world" probabilities). We also argue that, though probability scenarios may look appealing to foster investor protection, their practical implementation, if based on the right economic approach, raises significant regulatory and enforcement problems.

JEL Classifications: C02, C51, C58, G12, G17, G33.

Keywords: derivatives, structured products, risk-neutral pricing, probability scenarios.

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1. Introduction

In principle, an appealing way to enhance the regulation on the transparency of derivatives and structured products would be that of mandating the disclosure of the probability that their value at some future dates exceeds a given threshold (for example, the issue price increased by the risk-free rate). In the rest of the paper we will term this approach as "probability scenarios".

This work clarifies that it would be misleading to calculate probability scenarios using the probabilities underlying the pricing process, because these probabilities assume that market participants are risk-neutral (so-called "risk-neutral" probabilities). In reviewing the relevant mathematical finance literature, we argue that this approach is economically grounded for pricing purposes, while it is highly questionable to estimate future values of derivatives.

In the case of equity derivatives, we can assume that the risk premium is zero because, as originally highlighted by the seminal work of Black and Scholes on option pricing, the derivative's price should be independent of investors' risk aversion, as long as the derivative is replicable with the underlying asset. However, the situation is completely different when we want to make inference on the future value of derivatives (i.e. for risk management purposes). In this case, in order to simulate the future value of the underlying, it does not make sense to assume that stock prices grow at the risk-less rate, as is instead acceptable for pricing. The growth rate should be equal to the risk-less rate plus the equity risk premium.

In the case of interest-rate derivatives, a zero risk premium is equivalent to assuming that forward rates are unbiased predictors of future rates (this is the so-called "pure expectations theory" for the term structure of interest rates). This means that investors require no compensation for the risk of unpredicted changes in future rates when buying long-terms bonds.

In summary, we argue that risk-neutral probabilities are acceptable for pricing, but not to forecast the future value of an asset.

Hence, probability scenarios should be calculated assuming that investors demand a risk premium to hold risky assets, such as shares or long-term bonds (so-called "real-world" probabilities).

The difference between risk-neutral and real-world probability is obviously well known in the academic debate. In this work we review the relevant mathematical finance literature in order to make clear the economics underlying our argument. In section 2 we discuss the issue for equity derivatives, while in the next section we move on to interest-rate derivatives (IRD); boxes contain the more technical material, and can be skipped by more policy-oriented readers. In the final section, we argue that, even if mandatory regulation on probability scenarios were correctly based on real-world probabilities, there would still be considerable regulatory and enforcement problems.

2. Equity derivatives

In the following §2.1, we briefly review the risk-neutral pricing theory. We rely on a standard illustration that can be found in any advanced text of stochastic calculus, trying to make clear the economics behind the mathematical modeling, which sometime is overlooked in the policy debate. The limits of the risk-neutral approach for probability scenarios will be then discussed at the technical level in §2.2.

2.1 The risk-neutral approach in the pricing of equity derivatives

Stochastic models developed in the mathematical finance literature allow to simulate the dynamics of equity prices and hence of future payoffs of any equity derivative or a structured product. Hence, it is possible to calculate the probability distribution of the value of a derivative at a certain future date T. The expected value in T is then discounted, and this gives the "fair value" or "theoretical price" of the derivative.

The key issue here is to understand the hypotheses behind the stochastic models used to simulate future stock prices. One possible model to simulate stock prices is the so-called "geometric Brownian motion", i.e. a stochastic differential equation that describes the change in share price S in an infinitesimal time interval as:

$$dS_t = \mu S_t dt + \sigma S_t \, dW_t \tag{1}$$

where dS_t is the price change, μ is the so-called *drift* of the process (whose economic interpretation is the expected return on equity, i.e. the risk-less rate plus the equity risk premium), σ is the expected volatility of the share price (also known as the "*diffusion*" parameter of the process), W_t is a random variable, so that ΔW in Δt is equal to $\varepsilon * \sqrt{\Delta t}$, where ε is a standard normal distribution (i.e. zero mean and unit variance), and such that the values of ΔW in any two intervals Δt are independent.

Equation (1) basically assumes that equity returns (dS_t/S_t) have a normal distribution with average $\mu\Delta t$ and standard deviation $\sigma\Delta t$, and that returns are independent over time (and hence assumes the informational efficiency of markets). Equation (1), therefore, models equity returns (dS_t/S_t) as the sum of a deterministic component proportional to the value of the parameter μ and of a random component σdW_t which generates random increases/decreases that are, however, i.i.d..

The crucial point in using the equation (1) to simulate future values of stock prices is the estimate of μ. In fact, while the volatility σ is normally set to the implied volatility quoted in option markets, μ needs an explicit assumption on the equity risk premium, which is obviously a quite complex task and the economic literature on the subject is huge. Though the issue of surveying such literature is beyond the scope of this paper (but we will come back on this at the end of §2.2), it is quite clear that different models can be used and results can differ substantially (even when the same model is estimated over different dataset).

However, it can be shown that (for pricing only), under standard conditions of complete and frictionless markets with no arbitrage opportunities, future share prices can be simulated using the following equation:

$$dS_t = rS_t dt + \sigma S_t \, dW_t \qquad (2)$$

where the expected return on equity μ (i.e. the equity premium plus the risk-free rate) is replaced by the risk-free rate r. This result has obviously a great importance because it postulates that derivative prices are independent of investors' risk aversion and the original intuition comes from the work of the early seventies of the last century by the economists Robert Merton, Myron Scholes and Fischer Black, which was worth them the Nobel prize in 1997.

The economic intuition is, in fact, very simple and can be summarized as follows: if the payoff of a derivative is replicable with a portfolio composed by the underlying asset and by a risk-free security (so-called "*replicating portfolio*"), then the price of the derivative and that of the replicating portfolio must be the same, otherwise profitable arbitrages would arise. The no-arbitrage principle leads directly to the conclusion that the price of the derivative does not depend on the risk aversion of market participants, and hence on the risk premium required to hold shares.

In Box 1 we present a simple example to show how derivatives can be priced without making any hypotheses on the risk premium, which also serves as a preliminary technical introduction to risk-neutral probabilities.

Box 1 − *Risk-neutral* pricing in a simplified case with 2 possible states of the world

Consider a call option that expires in *t+1*. The value in *t+1* of the option (*C*) depends on the share price (*S*) in *t+1*, which has a binomial structure. In the binomial model, it is assumed that trades occur at discrete instants *t, t+1, t+2, ...* and that the price *S* of the underlying share follows a binomial multiplicative stochastic process. This means that at the end of each period the price of the underlying share is given by its starting value multiplied by a factor *a* or *b*, with *a* and *b* being real positive values, known and constant in all periods. For simplicity, the values at the time *t+1* are summed up in the following framework:
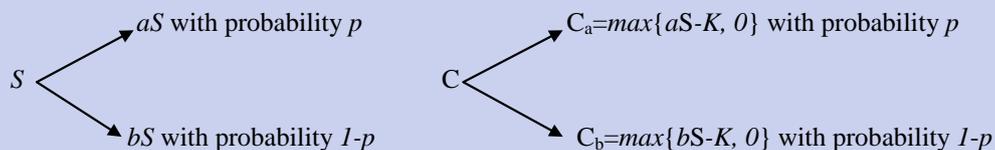
Therefore the equity return in the generic period [*t+j, t+j+1*] assumes the value *a* with probability *p* and the value *b* with probability *1-p*.

It is assumed that $a > b$, so that aS indicates returns higher than in bS. It is supposed that the share does not pay dividends and that the usual hypotheses of perfect markets hold (short sales allowed and the market is frictionless; agents are price-takers and profit-maximizing; risk-free arbitrage is absent). It is also supposed that future *risk-free* rates

are constant and known in t:

$$i(t+j, t+j+1)=i.$$

Therefore, $m=1+i$ is the *riskless* value of an annuity in the generic period $[t+j, t+j+1]$. It can be shown that in order to avoid profitable risk-free arbitrage it must be $a > m > b$.

Suppose we make a portfolio composed of Δ shares and an investment of B euro at the *risk-free* rate i. The value of this portfolio will be:
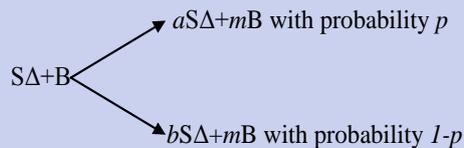
Cox, Ross and Rubinstein (1979) have shown that it is possible to calibrate the weight Δ of the share component and the weight B of the bond component of the portfolio in order to exactly replicate the payoff of the call option, so that it assumes the value C_a if the market rises and C_b if the market falls, i.e.:

$$\begin{cases} aS\Delta + mB = C_a \\ bS\Delta + mB = C_b \end{cases} \qquad [\text{A.1}]$$

The solutions of the system of two linear equations are:

$$\Delta = \frac{C_a - C_b}{(a-b)S} \qquad [\text{A.2}]$$

and

$$B = \frac{aC_b - bC_a}{(a-b)m}. \qquad [\text{A.3}]$$

To avoid arbitrage, the price of the call option in t must therefore be equal to the price of the replicating portfolio in t, i.e.:

$$C = S\Delta + B. \qquad [\text{A.4}]$$

This last equation represents an option pricing formula that it is totally independent from the subjective expectations on p, and therefore it implies that even if market participants have different opinions on the probabilities of future events and different risk tolerance, they still must agree on the same price for the call option.

In fact, by replacing the expressions shown for Δ and B, the price C can be expressed as:

$$C = \frac{C_a - C_b}{a - b} + \frac{aC_b - bC_a}{(a-b)m} = \frac{1}{m}\left(\frac{m-b}{a-b}C_a + \frac{a-m}{a-b}C_b\right);$$ [A.5]

If we define:

$$q = \frac{m-b}{a-b},$$ [A.6]

(A.5) can be written as:

$$C = \frac{1}{m}[qC_a + (1-q)C_b].$$ [A.7]

The option price can now be interpreted as the discounted expected value that the option may have in two future states of the world weighted with a new probability measure *q* e *1-q*. These probabilities are known as *risk-neutral*, because they do not depend on the subjective probability *p* of the occurrence of the possible states of the world, nor on the risk aversion of market participants.

Hence, under *q* and *1-q*, the value of a derivative instrument can be expressed not only as the arbitrage-free price, but also as the discounted expected payoff at maturity. In fact, it can be demonstrated that:

$$C = S\Delta + B = \frac{1}{m}[qC_a + (1-q)C_b]$$

This is the so-called equivalent martingale measure, as defined in Harrison and Kreps (1979), which states that by transforming *p* into *q* we redistribute the probability mass so that the no-arbitrage price of a financial asset (left-side part of the equation) is equal to its future discounted value at the risk-free rate (right-side part of the equation)[4].

We can check all this with a numeric example. Suppose in *t=0* S_0 =10 and that in *t=1* S_a is equal to 14 (*a*=1.4) and S_b to 6 (*b*=0.6); suppose also that the risk-less rate *i* is equal to 10% and the strike price is 10; the no-arbitrage price of the call option is determined as follows:

a) we first determine the value in *t=1* of the call option in the two states of the world:

$$C_a = max\,(0;\,S_a\text{-}K) = max\,(0;\,14\text{-}10) = 4$$

$$C_b = max\,(0;\,S_b\text{-}K) = max\,(0\;6\text{-}10) = 0$$

b) then by using equations [A.2] [A.3] [A.4] we have the *no-arbitrage* price:

$$\Delta = \frac{C_a - C_b}{(a-b)S} = \frac{4-0}{14-6} = 0.50$$ [A.8]

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4   The demonstration that the formula of the expected value is martingale equivalent to the no-arbitrage price is contained in Baxter and Rennie (1996).

$$B = \frac{aC_b - bC_a}{(a-b)m} = \frac{1.4*0 - 0.6*4}{(1.4-0.6)1.1} = -2.72 \qquad \text{[A.9]}$$

$$C_0 = S_0\Delta + B_0 = 10 * 0.50 - 2.72 = 2.29 \qquad \text{[A.10]}$$

We can see that in $t=1$ the replicating portfolio will have a value of either $S_a\Delta + B_1$ (14*0.50-3=4) or $S_b\Delta + B_1$ (6*0.50-3=0), exactly like the call option. We can also check that the *no arbitrage* price is equal to the discounted expected value of the payoffs according to the new "probability measure" q defined in equation A.6:

$$q = \frac{m-b}{a-b} = \frac{1.1-0.6}{1.4-0.6} = 0.625 \cong 0.63 \qquad \text{[A.6 bis]}$$

$$C = \frac{1}{m}[qC_a + (1-q)C_b] = \frac{1}{1.1}[0.63 * 4 + (1 - 0.63) * 0] = 2.29$$

The above results can be extended to any derivative with payoff $Y(t+1) = f[S(t+1)]$ where f is the contractually specified function that links the value of the derivative to the value of the underlying asset S. The conditions to be satisfied for the stock-bond portfolio to replicate the payoff produced by the derivative are expressed by the linear system in the unknowns Δ and B:

$$\begin{cases} aS\Delta + mB = Y_a \\ bS\Delta + mB = Y_b \end{cases}$$

The no-arbitrage rule requires that the price Y in t be equal to the value in t of the replicating portfolio, thus giving the pricing formula:

$$Y = S\Delta + B. \qquad \text{[A.11]}$$

It is possible to show that using the risk-neutral probability measure q:
$$q = \frac{m-b}{a-b},$$

the pricing equation can be written as:

$$Y = \frac{1}{m}[qY_a + (1-q)Y_b]. \qquad \text{[A.12]}$$

The pricing formula [A.12] has a great relevance because it does not depend on the subjective (real-world) probability p and it does not require any hypothesis on market participants' risk aversion.

We now discuss how this result relates to the classical expected utility hypothesis, which assumes risk averse agents.

If the representative agent is profit maximizing and risk-averse, with a monotonic and concave utility function $u(x)$, the criteria to value an uncertain future payoff $Y(T)$ is based on discounting back at the risk-free rate i the certainty equivalent:

$$\bar{\bar{Y}}(t + 1) = u^{-1}\{E_t[u(Y(t + 1))]\} = u^{-1}[pu(Y_a) + (1 - p)u(Y_b)].$$

With this approach, we would therefore have:

$$Y = \frac{1}{m}u^{-1}[pu(Y_a) + (1 - p)u(Y_b)],$$

or, with compact notation:

$$Y(t) = \frac{\bar{\bar{Y}}(t+1)}{m}. \qquad\qquad [A.13]$$

Given risk-aversion, the certainty equivalent $\bar{\bar{Y}}(t + 1)$ cannot be greater than the expected value of $Y(t+1)$ and therefore:

$$\bar{\bar{Y}}(t + 1) \leq E_t[Y(t + 1)],$$

or:

$$u^{-1}[pu(Y_a) + (1 - p)u(Y_b)] \leq pY_a + (1 - p)Y_b.$$

This inequality is a direct consequence of the concavity of the $u(x)$. If one excludes the trivial case $Y_a = Y_b$ (corresponding to a deterministic payoff) a certainty equivalent and the expected value of the lottery can be the same only if the utility function is linear; i.e. only a risk-neutral agent will price the derivative according to the rule:

$$Y(t) = \frac{E_t[Y(t+1)]}{m}. \qquad\qquad [A.14]$$

Since it must be that $a>m>b$ in order to avoid risk-less arbitrages, the coefficient q in equation [A.12] is bounded between 0 and 1, and it can therefore be interpreted as a *pseudo probability*. If one accepts this interpretation, the expression in square brackets in [A.12] can be interpreted as the expect value in t of the uncertain payoff $Y(T)$, based on the pseudo-probabilities q and $1-q$; i.e. it can be represented as:

$$E_t^Q[Y(t + 1)] = qY_a + (1 - q)Y_b \qquad\qquad [A.15]$$

where E_t^Q represents the expectation calculated in t according to the probability q. The price of the derivative in t can therefore be expressed in the form:

$$Y(t) = \frac{E_t^Q[Y(t+1)]}{m} \qquad\qquad [A.16]$$

Hence, [A.16] and [A.14] show that the pricing formula based on the no-arbitrage rule is equivalent to the pricing criteria of a risk-neutral agent[5]. Of course, the key point here is that [A.16] does not imply that market participants are actually risk-neutral. The real meaning of the adjustment for the risk produced by the probability q is evident if one compares [A.13] with [A.16]: the risk-neutral

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[5]   For a formal but accessible demonstration, see Baxter and Rennie (1996).

expectation $E_t^Q[Y(t+1)]$ plays the role of a certainty equivalent, but the risk adjustment is not obtained by using the utility function. Hence, the shape of the risk-neutral probabilities, and therefore the distortion of the weights in the calculation of the expected value, will be the same for all derivatives with the same underlying S and will depend only on the dynamics of the underlying S and the risk-free rate.

It can be concluded, therefore, that if there is agreement between agents on the parameters a and b that specify the dynamics of the process S, a derivative on S must have the same price for all agents, regardless of the subjective probability p and of the individual utility function $u(x)$. In short, the pricing method according to the no-arbitrage principle ensures that any risk-averse agents will value the derivative "as if" he were risk-neutral.

Extension to a multi-period framework

Using the concept of martingale[6] it is possible to extend CRR model to the valuation of a generic derivative on S in a multi-period binomial model. Using the no-arbitrage rule previously discussed, it is possible to show that the price of a generic derivative with payoff $Y(t+n) = f[S(t+n)]$ (for which early exercise is not allowed) is:

$$Y = \frac{1}{m^n}\left[\sum_{k=0}^{n}\frac{n!}{k!(n-k)}q^k(1-q)^{n-k}f(a^k b^{n-k}S)\right], \qquad \text{[A.17]}$$

since $f(a^k b^{n-k}S)$ is the value assumed by $Y(t+n)$ after k increases and $n-k$ decreases in the price S. The price of the derivative can also be represented in the usual form of discounted risk-neutral expectation:

$$Y(t) = \frac{1}{(1+i)^n}E_t^Q[Y(t+n)]. \qquad \text{[A.18]}$$

It is possible to show that same pricing formula holds at any time $\theta \leq T$, so that:

$$Y(\theta) = \frac{1}{(1+i)^{\tau-\theta}}E_\theta^Q[Y(\tau)], \quad t \leq \theta \leq \tau \leq T. \qquad \text{[A.19]}$$

If we define the discounted pricing process:

$$Y^*(\theta) = \frac{Y(\theta)}{(1+i)^{\theta-t}} \qquad \text{[A.20]}$$

we have:

$$Y^*(\theta) = E_\theta^Q[Y^*(\theta+1)]. \qquad \text{[A.21]}$$

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[6] Starting from the succession of a random variables X_n, it is possible to define the stochastic process in the discrete time $Y_n = Y_0 + \sum_{k=1}^{n}X_k$. The process Y_n is a martingale if $E_{n-1}(Y_n) = Y_{n-1}$. Since Y_{n-1} is known at the time $n-1$ then one can write $E_{n-1}(Y_n - Y_{n-1}) = 0$. Therefore a martingale can be intuitively featured as a process whose increases represent the payoff of a fair game.

In fact, from [A.20] we have:

$$E_\theta^Q[Y^*(\theta+1)] = E_\theta^Q\left[\frac{Y(\theta+1)}{(1+i)^{\theta+1-t}}\right] = \frac{1}{(1+i)^{\theta-t}}E_\theta^Q\left[\frac{Y(\theta+1)}{(1+i)}\right];$$

and from [A.19] one also has:

$$\frac{1}{(1+i)^{\theta-t}}E_\theta^Q\left[\frac{Y(\theta+1)}{1+i}\right] = \frac{Y(\theta)}{(1+i)^{\theta-t}} = Y_\theta^*.$$

[A.21] therefore states that the discounted pricing process of the derivative is a *martingale* when using risk-neutral probability measure. More generally, it can be shown that under the risk-neutral probability measure the discounted value of a derivative is a martingale:

$$\frac{Y_n}{(1+i)^n} = E_n^Q\left[\frac{Y_{n+1}}{(1+i)^{n+1}}\right], n = 0,1,\dots,N-1.$$

which gives the so-called "*First Fundamental Asset Pricing Theorem*", according to which:

$$E^Q\frac{Y_n}{(1+i)^n} = Y_0, n = 0,1,\dots,N \qquad\qquad [A.22]$$

This means that future increases of the discounted pricing process have always a zero expected value under the risk-neutral probability measure[7].

Box 1 has discussed in a simplified way the economic ideas behind the seminal work of Robert Merton, Myron Scholes and Fischer Black, pointing to their mathematical generalization, whereby it is possible to show that under the risk-neutral measure the discounted value of the expected payoff of a derivative has a martingale behavior. This means that the risk-neutral measure guarantees that the derivative price satisfies a no-arbitrage condition.

When there is no closed-end formula for derivative pricing, one needs to use simulations techniques. Here the problem is technically different, or in any case of a more general nature than that faced in the work of Black and Scholes (1973), since we need to demonstrate mathematically that it is possible to use the equation (2) instead of the equation (1) to simulate future payoffs. In fact, the work of Robert Merton, Myron Scholes and Fischer Black was later generalized from the mathematical viewpoint, showing that, under the mentioned conditions of complete and frictionless markets, it is possible to calculate the price of a derivative as the discounted expected value of future payoffs using a risk-neutral probability measure, rather than real probabilities. In practical terms, this means that it possible to use equation (2), instead of equation (1), to simulate future payoffs, and hence that it is possible

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[7] For a more complete discussion, see Baxter and Rennie (1996) and Shreve (2004).

to get rid of the big problem of the equity premium estimation. Equation (2) just needs estimates of the risk-free rate r and of the volatility σ, which can easily be taken from market quotes.

The fundamental concepts at the basis of this result are outlined below, while a formal analytical illustration given in Box 2.

As previously shown, the price at time t of a derivative product that generates uncertain payoffs X_T at time T is equal to the discounted expected value of future payoffs, i.e. $P_t = E_P[X_T] * K(t,T)$, where E_P indicates the expected value of the payoffs X_T under the probability measure P, while $K(t,T)$ is the discounting factor (i.e. the price of a zero-coupon bond).

It can be demonstrated that, under the assumption of complete markets with no-arbitrage, the same price is obtained using a risk-neutral probability measure Q, i.e. $P_t = E_Q\left[\frac{dP}{dQ} * X_T\right] * K(t,T)$, where E_Q indicates the expected value under the risk-neutral probability Q, $\frac{dP}{dQ}$ is the so-called Radon–Nikodyn derivative and $K(t,T)$ is again the discounting factor. Applying certain general results of probability theory it can be shown that the change from a real-world probability measure (P) to a risk-neutral probability measure (Q) can be obtained by just changing the drift of the stochastic process (and not of the volatility) of the underlying. These mathematical results formally show that it is possible to calculate the price of a derivative under the risk-neutral probability measure by just changing the drift of the stochastic process of the underlying. In short, these results allow, to use the equation (2), instead of the equation (1), to price equity derivatives.

More precisely, the Radon–Nikodyn derivative and the Girsanov theorem define the technical steps to change the probability measure, from real-world P to risk-neutral probability Q, when the stochastic process for the underlying is a geometric Brownian motion, so that the two probability measures become "equivalent" for the purposes of calculating expected values. However, this does not mean that for the same set of events the two probability measures assume equal values, nor that they have equal moments other than the expected value.

In summary, the mathematical results illustrated above do nothing more than generalize and formalize the original ideas of Black and Scholes, showing that, under complete markets with no arbitrage, it is possible to use for pricing purposes (only) stochastic models that do not factor in the equity premium.

Box 2 (based on material from standard textbooks of advanced stochastic calculus) gives a formal proof that equation (2) represents a change of probability measure in respect to equation (1), i.e. a change from a real-world probability measure (P) to a risk-neutral measure (Q), and that this change of measure ensures equivalence for pricing purposes (i.e. in terms of expected value).

Box 2 – Change of the probability measure from real-world to risk-neutral when the stochastic process is a geometric Brownian motion

We define P and Q as two probability measures out of a space of finite events Ω and we will assume that $P(w) > 0$ and $Q(w) > 0$ for every $w \in \Omega$. If we defined the random variable

$$Z(w) = \frac{Q(w)}{P(w)} \qquad \text{[B.1]}$$

it is possible to show that:

(i) $P(Z > 0) = 1$;
(ii) $E^P Z = 1$;
(iii) for every random variable Y we will have $E^Q Y = E^P[ZY]$, i.e. the expected value of the random variable Y under the probability measure Q is equal to the expected value under the probability P of the random variable Y transformed by the random variable Z.

Proof of (*iii*) can easily be found:

$$E^Q Y = \sum_{w \in \Omega} Y(w) Q(w) = \sum_{w \in \Omega} Y(w) \frac{Q(w)}{P(w)} P(w) = \sum_{w \in \Omega} Y(w) Z(w) P(w) = E^P[ZY].$$

In a continuum of events the change of probability measure from the real-world P to the risk-neutral Q is similarly obtained by defining a space of probabilities (Ω, F, P) and assuming that Z is a non-negative variable so that $E^P Z = 1$. Therefore, for every $A \in F$ we will have

$$Q(A) = \int_A Z(w) dP(w) \qquad \text{[B.2]}$$

i.e. Q is a new probability measure such that:

$$E^Q Y = E^P[ZY] \qquad \text{[B.3]}$$

where the random variable Z represents the *Radon-Nikodym* derivative of Q in respect of P, i.e.:

$$Z = \frac{dQ}{dP}. \qquad \text{[B.4]}$$

Let us now introduce the Girsanov theorem.

Girsanov Theorem: $W(t)$, with $0 \leq t \leq T$, is a Brownian movement defined on a space (Ω, F, P) and $\theta(t)$, with $0 \leq t \leq T$, is a generic stochastic process. If we define:

$$Z(t) = exp\left\{ -\int_o^t \theta(u) dW(u) - \frac{1}{2} \int_o^t \theta^2(u) du \right\} \qquad \text{[B.5]}$$

$$\widetilde{W}(t) = W(t) + \int_o^t \theta(u)du \qquad [B.6]$$

the result will be that, under the risk-neutral probability measure defined in the equation [B.2], the stochastic process $\widetilde{W}(t)$ is also a Brownian movement.

Now we take the geometric Brownian motion:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \le t \le T \qquad [B.7]$$

where the drift μ represents the expected return on equity and σ its volatility.

[B.7] can be written in an equivalent form as:

$$S(t) = S(0)exp\left\{\int_o^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}. \qquad [B.8]$$

Let us suppose, in addition, that we have a process which describes the interest rate $R(t)$. We can then write the discounting process as:

$$D(t) = e^{-\int_0^t R(s)ds} \qquad [B.9]$$

and by applying the Itô-Doeblim formula, we will have:

$$dD(t) = -R(t)D(t)dt. \qquad [B.10]$$

The discounted price process will therefore be:

$$D(t)S(t) = S(0)exp\left\{\int_o^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}$$

and its differential is:

$$d\big(D(t)S(t)\big) = \big(\alpha(t) - R(t)\big)D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t)$$
$$= \sigma(t)D(t)S(t)[\theta(t)dt + dW(t)] \qquad [B.11]$$

where we have defined the market price of the risk as:

$$\theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}.$$

The pricing problem at the basis of the original idea of Black and Scholes discussed in the text consists in changing to a new probability measure Q that excludes any arbitrage possibility. In order to do this it is sufficient to find the only martingale probability measure Q equivalent to P for the pricing of a contingent claim (Harrison and Kreps, 1978).

We now apply to the Girsanov theorem, which establishes that under the probability measure Q the stochastic process $\widetilde{W}(t)$ defined in [B.6] is Brownian motion. We can then rewrite [B.11] in the terms of the stochastic process $\widetilde{W}(t)$ as:

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\widetilde{W}(t). \qquad\qquad \text{[B.12]}$$

The probability measure underlying the equation [B.12] is, as mentioned, the risk-neutral probability measure equivalent to the real-world probability measure that makes the discounted price process $D(t)S(t)$ a martingale.

The original non-discounted process $S(t)$ has therefore an expected return equal to the risk-free rate under the risk-neutral probability measure Q. In fact, it can easily be verified that, replacing

$$dW(t) = -\theta(t)dt + d\widetilde{W}(t) \text{ [obtained from B.6]}$$

in [B.7] gives:

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t). \qquad\qquad \text{[B.13]}$$

Hence, the change of probability measure has allowed us to set the equity premium to zero and hence to replace μ with the risk-free rate r. Since r can be uniquely defined, Q represents the only equivalent martingale measure for S. This obviously does not mean that the real drift of equity returns is the risk-less rate. The new probability measure Q is just a mathematical expedient to make the pricing problem tractable and come up with a price that does not depend on risk aversion as in the Black and Scholes models.

2.2 Why risk-neutral probabilities cannot be used to forecast the future value of derivatives

In the previous section we have shown that, under standard assumptions of frictionless, complete and efficient markets, it is possible to price derivatives as the discounted value of expected future payoffs assuming that the equity premium is zero.

It has been shown that this approach is acceptable for pricing purposes only, because, under the risk-neutral probability measure, the discounted value of expected future payoffs has a martingale behavior that satisfies no-arbitrage conditions, and this allows to ignore the risk premium. If instead we are not taking discounted values, risk-neutral probabilities cannot be used to estimate the (future) value of derivatives, because these probabilities would not guarantee a no-arbitrage condition.

This means that we cannot use risk-neutral probability to value derivatives at future dates. Hence, probability scenarios on future values need to be calculated using probabilities that factor in the equity premium (so-called "real-world" probabilities), because the representative agent is likely to be risk-averse. In other words, in order to calculate probability scenarios for future values, it is necessary to simulate payoffs using equation (1), while equation (2) is instead valid for pricing only.

Probability scenarios are essentially risk management tools widely used by financial intermediaries. The so-called value-at-risk (VaR) is just an example of a particular way to exploit the information given by probability scenarios. In fact, VaR gives the expected loss corresponding to a given percentile of the probability distribution of the value of the financial product at a certain future date. Therefore, VaR calculation requires the estimation of the entire probability distribution of the value of a financial product at a given future date, exactly like the in the probability scenarios. Box 3 gives an example that shows how the application of VaR models requires real-world probabilities.

Box 3 – VaR calculation and real-world probabilities

Suppose we have a structured product whose price P depends in a non-linear way from the price S of a share. The change in the price P can be estimated applying the Taylor-rule approximation:

$$\Delta P = S\delta\Delta x + \frac{1}{2}S^2\gamma(\Delta x)^2$$

where δ and γ are respectively the first and second derivative of the price of the derivative product with respect to S, while $\Delta x = \frac{\Delta S}{S}$ is the stock return. ΔP does not have a known distribution, but if we assume that Δx has a normal distribution (if S follows a geometric Brownian motion), VaR can be calculated extracting values of Δx from a normal distribution with given mean and the variance. If the mean is estimated on historical data, it will include an estimate of the risk premium, and this means that the probability distribution of ΔP is not risk-neutral.

When, especially for regulatory purposes, VaR is used to evaluate risks over very short time periods (typically from one to very few days), it is obviously possible to set the risk premium to zero without making any significant errors, and thus VaR will reflect just volatility. This approach, however, is not correct, as shown above, when risks are valued over longer time periods.

The issue of the pitfalls in risk-neutral distributions is well known in finance literature. For example, Grundy (1991) underlines that risk-neutral probabilities are not "true" real-world probabilities, and by comparing the two distributions he derives an estimate of the investors' risk aversion. Similarly other papers try to correct risk-neutral probabilities derived from option prices in order to obtain real-world probabilities, which give a more accurate representation of the expectations of risk averse investors (Liu *et al.*, 2007 and Humphreys and Noss, 2012). More explicitly, Bliss and Panigirtzoglou (2004) clearly state the problem: *"Unfortunately, theory also tells us that the PDFs* [probability distribution functions – Ed. note] *estimated from option prices are risk-neutral. If the representative investor who determines option prices is not risk-neutral, these PDFs need not correspond to the*

representative investor's (i.e. the market's) actual forecast of the future distribution of underlying asset values".

The simple example in Box 4 shows more formally, using the example in Box 1, why risk aversion makes risk-neutral probabilities different from real-world probabilities.

Box 4 – The difference between *risk-neutral* and *real-world* probabilities in a framework with two 2 possible states of the world and risk-averse agents

Let us take the model of Box 1 where a derivative price Y depends on the underlying asset S, which in turn can assume only 2 values corresponding to 2 different states of the world a and b. We will now analyze the utility that a representative risk-averse agent (i.e. with a concave utility function) assigns to the value of the underlying share at the maturity date of the derivative T. The agent maximizes the utility of his final wealth $U(S)$ at time T. The representative agent maximizes the expected wealth on the basis of real-world probabilities p and 1-p associated with the 2 states of the world, i.e.:

$$\max p\, U(S_a) + (1 - p)U(S_b). \qquad [\text{E.1}]$$

However, the maximization process must take into account the restriction that, for the no-arbitrage argument discussed in Box 1, the price S of the underlying share at the future date T, i.e. the price of the forward derivative contract on the underlying share, must be equal to the risk-neutral price, i.e. $(qS_a + (1 - q)S_b)$. The current price of the share, i.e. the agent's initial wealth, must therefore necessarily be equal to the expected discounted value of the payoffs of the forward contract in T weighed for the risk neutral probabilities, i.e. as seen in Box 1:

$$e^{-rt}(qS_a + (1 - q)S_b) = S_0 \qquad [\text{E.2}]$$

Hence, to solve the constrained maximization problem we set the *Lagrange* function and solve for first order condition, as follows:

$$L = pU(S_a) + (1 - p)U(S_b) - \lambda(e^{-rt}(qS_a + (1 - q)S_b) - S_0)$$

$$\begin{cases} \dfrac{\partial L}{\partial S_a} = \dfrac{p\partial U(S_a)}{\partial S_a} - \lambda q e^{-rt} = 0 \\ \dfrac{\partial L}{\partial S_b} = \dfrac{(1 - p)\partial U(S_b)}{\partial S_b} - \lambda(1 - q)e^{-rt} = 0 \end{cases}$$

after some computations, we obtain:

$$p = \frac{\frac{\partial U(S_b)}{\partial S_b}}{\frac{\partial U(S_a)}{\partial S_a}\left(\frac{1}{q} - 1\right) + \frac{\partial U(S_b)}{\partial S_b}}$$

This expression shows how real-world probabilities and risk-neutral probabilities are linked through the utility function of economic agents. With a risk-neutral utility function, such as a linear one, real-world probabilities are equal to risk-neutral probabilities. For example, if:

$$U(S) = \alpha(S_a + S_b)$$

$$p = \frac{\alpha}{\frac{\alpha}{q} - \alpha + \alpha} = q$$

With a risk-averse utility function, such as a logarithmic function, real-world probabilities are different from risk-neutral probabilities. In fact, if:

$$U(S) = lnS_a + lnS_b$$

$$p = \frac{S_a}{\frac{S_b}{q} - S_b + S_a}$$

If we set $\frac{S_a}{S_b} = w$, we can write:

$$p = \frac{w}{\frac{1}{q} - 1 + w} = q\frac{w}{1 - q + wq} \qquad [E.3]$$

Here p is always higher than q for any 0<q<1. Using the formula [E.3] − which links real-world probabilities p to the "pseudo probabilities" q − it is possible to measure the difference between p and q depending on agents' risk aversion, using the case of the pricing of a simple call option in Box 1.

In Box 1 we have determined the price of a call option, taking the case of a binomial model for one period and applying the no-arbitrage pricing principle (i.e. risk-neutrality). We obtained both the price of the call (equal to 2.29, from equation A.10) and the risk-neutral probabilities, equal to q=0.63 and 1-q= 0.37 (equation A.6 *bis*).

Now, removing the risk-neutrality hypothesis and assuming that the representative agent has a logarithmic utility function, we can easily move from risk-neutral probabilities to real-world probabilities. Using [E.3] for a risk-averse agent, given the risk-neutral probability q=0.63 and the underlying process for

stock price described by the binomial model S_a=14 and S_b=6 described in Box 1, one obtains that the real-world probability is equal to p=0.80. The distortion in the extraction of the probabilities under the risk-neutrality hypothesis is therefore highly significant and is graphically represented in Figure 1.

Figure 1 – *Real-world* probabilities and *risk-neutral* probabilities in 2 different states of the world with risk-averse agents with a logarithmic utility function
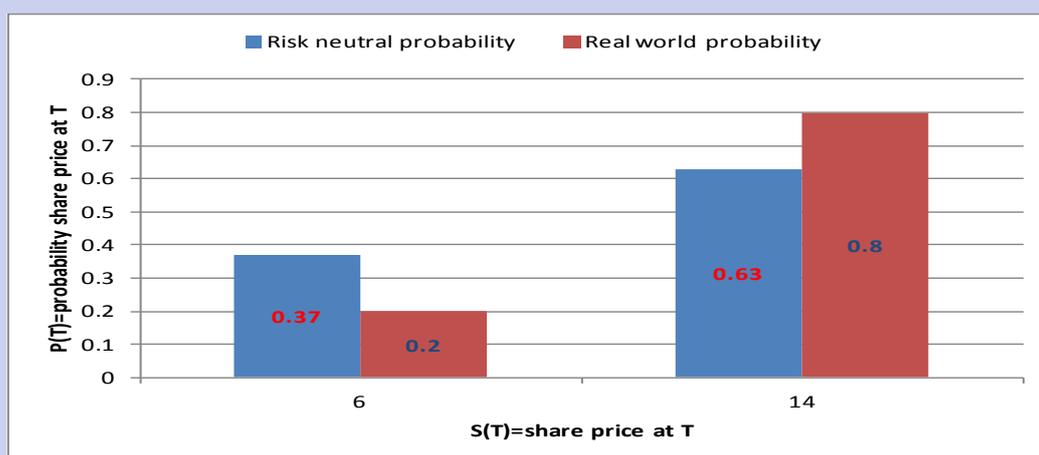
Hence, from the above discussion, it should be clear that only real-world probabilities can give meaningful measures of the likelihood that the value of a structured product at future dates will be higher or lower than a given threshold, because these probabilities are based on the plausible assumption that investors are risk-averse.

However, this introduces a significant complication in the practical computation of probability scenarios, because one needs to estimate the equity premium. This has been the subject of a huge strand of literature, which would be impossible to fully survey here. We just recall that there might be at least two options, one based on historical data and another based on a specific model of equilibrium expected returns (such as the CAPM and its subsequent modifications). The literature on equity premium estimates based on historical data starts with the original work of Siegel (1994) and more recent evidence have been provided, for example, by Fama and French (2001) and Dimson et al. (2003). All of these works show a significant variation of equity premium estimates through time and across countries. On the other hand, the evidence on the performance of CAPM (and its variation such as the Fama-French three factor model) tend to show unstable or mixed results (see Fama and French 2003, Ang and Chen 2003, Campbell and Vuolteenhao 2004, Hou et al. 2011 and Cochrane 2011).

In a recent paper, Ross (2014) develops a framework to tackle the issue of non-arbitrary estimates of real-world probabilities. In particular, Ross develops a method to jointly estimate the risk premium and the real-world probabilities from market prices (the so-called "recovery theorem"). In principle, as previously illustrated, a given risk-neutral probability can be compatible with a potentially infinite set of combinations of risk premium and real-

world probability (because risk-neutral probability is the product of risk premium and real-world probability). However, Ross shows that, under certain (non-parametric) technical assumptions on the stochastic processes and given a perfect knowledge on "state prices" (i.e. forward market prices conditional any possible future state of the world), it is possible to disentangle jointly the risk premium and the real-world probability from market prices. Hence, the "recovery theorem" would make real-world probabilities non-arbitrary, since they would be estimated from market prices only, without any assumption on the risk premium. However, the key drawback for a practical implementation of the recovery theorem, as noted by the same Ross, is that it requires the knowledge of the price of contingent forward contracts, i.e. the prices at future dates conditional on being in any other state of world. Even if we did have such set of information, there would be still scope for different implementation of the recovery theorem, so that it could be possible to come up with different real-world probabilities for the same product.

In summary, probability scenarios, if correctly based on real world-probabilities, are inevitably arbitrary or non-unique, since they rely on a specific model or dataset to estimate the equity premium. Thus, it is highly likely that the same derivative will have different probability scenarios, depending on the intermediary that is selling it.

In the conclusions we will discuss in more details the potential regulatory problems that would still arise even if probability scenarios were correctly based on real-world probabilities.

3.   Interest-rate derivatives

Risk-neutral pricing may apply to interest-rate derivatives (IRD) as well, though there are some important technical and economic differences with respect to equity derivatives that it is worth discussing.

In fact, IRD pricing is conceptually different from that of equity derivatives, since it requires explicit assumptions on the risk preference of market participants, while the pricing of equity derivatives is a "subordinated pricing", meaning that it only depends on the price of the underlying and does not require any assumption on investors' risk tolerance.

The basic reason for this is that IRD pricing depends on the stochastic process to model the so-called instantaneous spot rate, but the spot rate is not observable on the market. Nonetheless, using a no-arbitrage argument, and assuming a given risk premium, it is still possible to come up with a single price for any IRD. Moreover, differently form equity derivatives, in the IRD framework the assumption of a zero risk premium is underpinned by a specific economic theory on the term structure of interest rates (the "pure expectations theory", whereby current forward rates are assumed to be unbiased predictors of future rate), while in the equity derivatives setting risk-neutrality is simply a mathematical transformation supported by the no-arbitrage argument with no economic interpretation. However, assuming the pure expectations theory could be in any case arbitrary, since there are other alternative models on the term structure of interest rates, such as those that assume a liquidity premium, a preferred habitat or segmented markets, which imply positive or negative risk premia. Moreover, most of the empirical research on the US market tends to

reject the pure expectations theory (see Fama and Bliss 1987, Campbell and Shiller 1991, Cochrane and Piazzesi 2005).

In Box 5 we formalize the above discussion using material from standard textbooks of stochastic calculus.

Box 5 – The risk premium and the arbitrage argument for pricing of IRD

Let $Y(t)$ be the price in t of a generic IRD (or more generally, of any interest rate sensitive contract, which could be a floating rate note or zero-coupon bond) and $r(t)$ the instantaneous *spot rate*. The spot rate is modeled with a specific stochastic process and is basically the only source of risk in the pricing of IRD. We also assume that all agents agree on the following stochastic process for r:

$$dr(t) = f(r_t, t)dt + g(r_t, t)dZ(t)$$

where the *drift* $f(r_t, t)$ and diffusion process $g^2(r_t, t)$ need to be specified and $Z(t)$ is a Wiener process with unit variance. In this setting the $Y(t)$ will be a function of r and t:

$$Y(t) = Y(r_t, t)$$

with a similar stochastic process:

$$dY(t) = a(r_t, t)dt + b(r_t, t)dZ(t)$$

that we can rewrite as:

$$\frac{dY(t)}{Y(t)} = a'(r_t, t)dt + b'(r_t, t)dZ(t) \qquad \text{(F.1)}$$

where:

$$a'(r_t, t) = \frac{a(r_t, t)}{Y(t)}, \ \ b'(r_t, t) = \frac{b(r_t, t)}{Y(t)}.$$

The expected value of (F.1) is:

$$E_t\left[\frac{dY(t)}{Y(t)}\right] = a'(r_t, t)dt$$

so that:

$$a'(r_t, t) = \frac{E_t[dY(t)/Y(t)]}{dt};$$

and:

$$Var_t\left[\frac{dY(t)}{Y(t)}\right] = b'^2(r_t, t)Var_t[dZ(t)] = b'^2(r_t, t)dt$$

$$b'^2(r_t, t) = \frac{Var_t[dY(t)/Y(t)]}{dt}.$$

Hence a' e b' are the expected value and variance of the instantaneous return of the IRD, and the price per unit of risk is:

$$\pi_Y = \frac{a'-r}{b'}.$$

Suppose now we have a portfolio of one unit of and IRD with price $Y_1(t)$ and α unit of an IRD with price $Y_2(t)$. The value $W(t)$ of the portfolio is:

$$W(t) = Y_1(t) + \alpha Y_2(t)$$

and its stochastic process is:

$$dW = (a_1 + \alpha a_2)dt + (b_1 + \alpha b_2)dZ$$

where $a_k(r_t, t)$ and $b_k(r_t, t)(k = 1,2)$ are the coefficient for Y_k. For the portfolio share:

$$\alpha^* = -\frac{b_1}{b_2}$$

the value of W is:

$$W^* = Y_1 + \alpha^* Y_2,$$

and the dynamic is:

$$dW^* = (a_1 + \alpha^* a_2)dt. \tag{F.2}$$

Hence such portfolio is instantaneously risk-less. In order to avoid arbitrages, the change $dW^*(t)$ in dt has to be equal to the spot rate:

$$dW^*(t) = W^*(t)r(t)dt. \tag{F.3}$$

From (F.2) and (F.3) we get:

$$\frac{a_1'-r}{b_1'} = \frac{a_2'-r}{b_2'}. \tag{F.4}$$

Since the two IRD have been arbitrarily chosen, equation (F.4) implies that $(a_k' - r)/b_k'$ has to be same for all IRD, in order to avoid risk-less arbitrages. Hence, the function:

$$q(r_t, t) = \frac{a'(r_t,t)-r(t)}{b'(r_t,t)} \tag{F.5}$$

defines the characteristic function of the market in terms of the price required by market participants per unit of risk, i.e. the risk premium or "term premium", and it depends on $r(t)$ and t only. The key point here is that $a'(r_t, t)$ is not necessarily higher than $r(t)$, so that the term premium depends on investors' preferences in terms of inter-temporal wealth allocation. Since $r(t)$ is not a quoted rate, the term premium cannot be estimated on market data and the function $q(r_t, t)$ is then an exogenous element to the pricing model that needs to be specified, making some explicit assumptions on investors' preferences.

Once q has been defined, (F.5) can be rewritten as:

$$a - qb = rY$$

and substituting the functions for a and b, we have:

$$\frac{1}{2}g^2 \frac{\partial^2 Y}{\partial r^2} + (f - qg)\frac{\partial Y}{\partial r} + \frac{\partial Y}{\partial t} = rY. \tag{F.6}$$

(F.6) is a partial differential equation that represents the general pricing model of IRD. In order to solve (F.6) we need to specify the boundary conditions given by the specific type of IRD to be valued. The most simple valuation case is an IRD with a given payoff at maturity so that:

$$Y(T) = F(r_T) \tag{F.7}$$

where the price:

$$Y(t) = V[t; F(r_T)]$$

will be given from the solution to the general equation (F.6) under the condition that the value at maturity is given by (F.7). If the IRD pays no coupons, it can be shown that the solution to equation (F.6) under the constrain (F.7) has the following representation:

$$Y(t) = V[t; Y(T)] = \boldsymbol{E}_t^Q \left[e^{-\int_r^T r(u)du} Y(T) \right] \tag{F.8}$$

where $\boldsymbol{E}_t^Q$ is the expected value using the change in probability measure given by the following change in the drift in the spot rate:

$$\hat{f}(r_t, t) = f(r_t, t) - q(r_t, t)g(r_t, t)$$

Similarly to the pricing of equity derivatives, also in this setting the new probability measure Q can be interpreted as a risk-adjusted measure, since it takes into account the risk premium q. Note that here we use the term "risk-adjusted" and not "risk-neutral" simply because the risk premium q could be zero or negative, so that it has a different economic interpretation from the equity premium, which should always be positive. The other key difference from equity derivatives is that here a specific assumption on the risk premium is need in order to define Q, while in equity derivatives there is no need to define the risk premium to change the drift of the process.

Under the new probability measure Q, the expected value of any IRD over the period dt is equal to the spot rate:

$$\frac{E_t^Q[dY(t)]}{Y(t)dt} = r(t)$$

It can be shown that the assumption $q(r_t, t) = 0$ is equivalent to assuming the pure expectations hypothesis, whereby forward rates in t $i(t, T, s)$ are unbiased predictors of future rates $i(T, s)$. This means that we are assuming that economic agents do not require a compensation against the risk of unpredictable changes in future rates. Of course, under this hypothesis we have:

$$\hat{f}(r_t, t) = f(r_t, t)$$

so that the drift remains unchanged, and the *risk-adjusted* measure Q would be equal to the real-world measure P:

$$Y(t) = E_t\left[e^{-\int_r^T r(u)du}Y(T)\right] \tag{F.9}$$

This is however a quite special case. Any other theories on the term structure of interest rates would imply positive or negative risk premia, so that risk-adjusted probabilities will be different from real-world probabilities.

In order to clarify why the argument that risk-neutral probabilities are misleading for risk management is valid for IRD as well, we apply the theoretical framework described in the previous Box to a specific stochastic model for the spot rate developed by Cox, Ingersoll and Ross (1985) (CIR model).

In the CIR model f and g have the following specification:

$$f(r_t, t) = \alpha(\gamma - r_t) \quad \alpha, \gamma > 0; \tag{3}$$

$$g(r_t, t) = \rho\sqrt{r_t} \qquad \rho > 0. \tag{4}$$

where the drift defines a mean-reverting process and the diffusion process is proportional to $r(t)$, given the empirical evidence that volatility is increasing in the level of interest rates. Hence, the spot rate has the following stochastic process:

$$dr_t = \alpha(\gamma - r_t)dt + \rho\sqrt{r_t}dZ_t. \tag{5}$$

On the base of the mathematical arguments discussed in Box 5, it can be shown that for pricing purposes it is possible to use a risk-adjusted version of (5) and Cox, Ingersoll and Ross (1985) show that, from a general equilibrium model, the parameterization of the risk premium is the following:

$$q(r_t, t) = -\pi\frac{\sqrt{r_t}}{\rho} \tag{6}$$

where π is an arbitrary real number. Hence, the risk-adjusted version of (5) is:

$$\hat{f} = f - qg = \alpha(\gamma - r_t) + \pi r_t. \tag{7}$$

If we define:

$$\hat{\alpha} = \alpha - \pi, \quad \hat{\gamma} = \frac{\alpha}{\alpha - \gamma} \tag{8}$$

(7) can be re-written as:

$$\hat{f} = \hat{\alpha}(\hat{\gamma} - r_t)$$

The general valuation equation in Box 5 becomes:

$$\frac{1}{2}\rho^2 r\frac{\partial^2 Y}{\partial^2 Y} + [\alpha(\gamma - r) + \pi r]\frac{\partial Y}{\partial r} + \frac{\partial Y}{\partial t} = rY. \tag{9}$$

If we set $Y(t + \tau) = 1$ we can get a closed-hand formula for the term structure of zero-coupon discount rates:

$$v(t, t + \tau) = A(\tau)e^{-r(t)B(\tau)} \tag{10}$$

where A and e B are deterministic functions with the following form:

$$A(\tau) = \left[\frac{2de^{(\alpha-\pi+d)\tau/2}}{(\alpha-\pi+d)(e^{d\tau}-1)+2d}\right]^{v} \tag{11}$$

$$B(\tau) = \frac{2(e^{d\tau}-1)}{(\alpha-\pi+d)(e^{d\tau}-1)+2d} \tag{12}$$

where:

$$d = \sqrt{(\alpha-\pi)^2 + 2\rho^2} \tag{13}$$

$$v = 2\frac{\alpha\gamma}{\rho^2} \tag{14}$$

Hence the pricing formula for $v(t, t + \tau)$ given by equations (11)-(14) depends on all of four parameters α, γ, ρ e π that characterize the stochastic process for the spot rate, though some are aggregated in the form $\alpha - \pi$ and $\alpha\gamma$. If we term them as $\hat{\alpha}$ e $\hat{\gamma}$ we may re-write A and B:

$$A(\tau) = \left[\frac{2de^{(\hat{\alpha}+d)\tau/2}}{(\hat{\alpha}+d)(e^{d\tau}-1)+2d}\right]^{v}$$

$$B(\tau) = \frac{2(e^{d\tau}-1)}{(\hat{\alpha}+d)(e^{d\tau}-1)+2d}$$

where d e v are still given by equations (13) and (14).

For pricing purposes, it may not necessary to make explicit assumptions on q, because there are methods to estimate A and B using market quotes (so-called "calibration techniques") without being necessarily explicit on the value of π (and hence of q). This means, differently from equity derivatives, that the pricing process is not explicitly assuming risk-neutral agents (i.e. agent that demand a zero term premium).

On the other hand, the practical implementation of the CIR model to simulate future values of any IRD requires inevitably and empirical estimate of all the parameters α, γ, ρ and π and hence a specific assumption on the risk premium q.

In summary, from the above discussion it should be clear that, although the economics underlying the pricing of IRD is somewhat different from that of equity derivatives, probability scenarios on IRD, exactly like those on equity derivatives, require an explicit assumption on the risk premium. If the risk premium is set to zero, this means that the probability scenarios are based on the assumption that current forward rates are unbiased predictors of future rates. This implies that investors require no compensation for the risk of unpredicted changes in future rates when buying long-terms bonds.

Moreover, like in the case of equity derivatives, there can be different econometric and calibration techniques to estimate the parameters in *A* and *B*[8], so that the same IRD can have different plausible probability scenarios.

4. Conclusions

This work has reviewed the economics behind the mathematical finance literature on the pricing of equity and interest-rate derivatives in order to clarify that the probabilities used in the pricing process assume risk-neutral agents (though the concept of risk neutrality for equity derivatives is somewhat different from interest rates derivatives), and therefore these probabilities are of limited use for forecasting the value of derivatives at future dates: they do not convey the "real" probabilities of future events and could be potentially misleading if retail investors do not fully understand the implications of the assumption of risk-neutrality. To this end, it may be more accurate to use probabilities based on an explicit assumption on the risk premium (so-called real-world probabilities).

Hence, we argue that information given to retail investors on the probability that at some future dates the value of a derivative will be higher or lower than given thresholds (so-called "probability scenarios") would optimally explicitly reference real-world probabilities.

However, the proposed approach also has significant limitations. The use of real-world probabilities may entail significant problems.

First, real-world probabilities imply that the same product can have different probability scenarios, because of different plausible approaches to estimate the risk premium. Second, it may be difficult to have retail investors fully understand the hypothesis and caveat behind such approaches, in order to avoid that they blindly rely on them; in fact, retail investors should realize that, although scenarios are based on real-world probabilities, these are not "true" probabilities, in the sense of "frequentist probability", but just estimates that are highly model-dependent, and that, for this reason, probability scenarios should not be the sole driver of the decision process. Finally, intermediaries may use such models in order to make products look particularly attractive, while identifying such tailor-made utilization of models is necessarily very difficult, especially for non-technical retail investors.

For these reasons, we argue that, though probability scenarios are an appealing tool to foster investor protection, their application in practice would need to be approached with caution by investors and regulators alike.

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[8]   See Gourieroux and Monfort (2007).

BIBLIOGRAPHY

Ang, A. and J. S. Chen (2003), CAPM Over the Long-Run: 1926-2001, mimeo.

Baxter, M. and A. Rennie (1996), *Financial Calculus. An Introduction to Derivatives Pricing*, Cambridge University Press.

Black, F. and M. Scholes (1973), *The pricing of options and corporate liabilities*, Journal of Political Economy.

Bliss, R. and N. Panigirtzoglou (2004), *Option implied risk-aversion estimates*, Journal of Finance.

Brandt, M. and D. Chapman (2002), *Comparing Multifactor Models of the Term Structure*, Working Paper, Wharton School.

Brown, S. J. and P. H. Dybvig (1986), *The empirical implications of the Cox, Ingersoll, Ross theory of the term structure of interest rates*, Journal of Finance.

Campbell, J. Y. and R. J. Shiller (1991), *Yield spreads and interest rate movements: a bird's eye view*, Review of Economic Studies.

Campbell, J. Y. and T. Vuolteenaho (2004), *Bad Beta, Good Beta*, American Economic Review.

Cochrane, J. H. and M. Piazzesi (2005), *Bond Risk Premia*, American Economic Review.

Cochrane, J. (2011), *Presidential Address: Discount Rates*, Journal of Finance.

Cox, J. C., Ross S. A. and M. Rubinstein (1979), *Option pricing: a simplified approach*, Journal of Financial Economics.

Cox, J. C., J. E. Ingersoll and S. A. Ross (1985), *A Theory of the Term Structure of Interest Rates*, Econometrica.

Dimson, E., P. Marsh and M. Staunton (2003), *Global Evidence on the Equity Risk Premium*, Journal of Applied Corporate Finance.

Fama E. and R. Bliss (1987), *The information in long-maturity forward rates*, American Economic Review.

Fama, E. F. and K. R. French (2002), *The Equity Premium*, Journal of Finance.

Fama, E. F. and K. R. French (2003), *The Capital Asset Pricing Model: Theory and Evidence*, CRSP Working Paper.

Fama, E. F. and K. R. French (2006), *The Value Premium and the CAPM*, Journal of Finance.

Gisiger, N. (2010), *Risk-neutral Probabilities Explained*, working paper.

Gourieroux, C. and A. Monfort (2007), *Estimation of the Historical Mean-Reverting Parameter in the CIR Model*, mimeo

Grundy, B. D. (1991), *Option prices and the underlying asset's return distribution*, Journal of Finance.

Harrison, J. M. and D. M. Kreps (1979), *Martingales and Arbitrage in Multiperiod Securities Markets*, Journal of Economic Theory.

Harrison, J. M. and R. Pliska (1981), *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Processes.

Hou, K., G. A. Karolyi and B. Kho (2011), *What Factors Drive Global Stock Returns?*, Review of Financial Studies.

Humphreys, V. and J. Noss (2012), *Estimating probability distributions of future asset prices: empirical transformations from option-implied risk-neutral to real-world density functions*, Bank of England, Working Paper.

Liu, X., M. Shackleton, S. Taylor and X. Xu (2007), *Closed-form transformations from risk-neutral to real-world distributions*, Journal of Banking and Finance.

Ross, S. (2014), *The Recovery Theorem*, Journal of Finance, forthcoming.

Shreve, E. S. (2004), *Stochastic Calculus for Finance*, Springer.

Siegel, J. (1994), *Stocks for the Long Run*, Irwin Professional Publishing.